Nonlinear Eddy Current NDE and Theory Based on Vector Preisach Model in Rayleigh Regime

Norio Nakagawa

Center for Nondestructive Evaluation, Iowa State University
Ames, Iowa 50011, USA

Abstract. This paper presents a nonlinear theory of eddy current NDE response, applicable to EC measurements involving ferromagnetic materials. The theory treats the nonlinearity at the lowest nontrivial order. At this lowest order, the well-known Rayleigh relation exists, describing the non-linear hysteretic B-H relation, except that it is restricted to the case where the B and H fields are unidirectional and parallel to each other. Prototypical EC NDE problems require extension of Rayleigh’s formula for vector-valued B and H fields. The paper examines a vector Preisach model, and points out that the vector extension of the Rayleigh relation is obtainable by the choice of a constant weight function. The given Preisach-Rayleigh model is amenable to explicit harmonic analysis, through which NLEC problems are soluble in Born approximation.

Keywords: Eddy Current; Nonlinear Theory; Preisach-Rayleigh Model

PACS: 81.70.Ex, 81.40.Rs, 75.60.-d

INTRODUCTION

It is well known that impedance signals collected by eddy current (EC) instruments exhibit nonlinearity when applied to ferromagnetic samples. The impedance $Z(I)$ is essentially the complex-valued ratio between the output voltage $V$ and the drive current $I$, and hence stays $I$-independent when the inspection system is linear. A ferromagnetic sample can turn $Z(I)$ into $I$-dependent through its nonlinear response to the excitation $H$ field. In order to isolate this nonlinear response from the signal, we may conduct impedance measurements for varying current intensity $I$, and examine the following quantity $\alpha$, defined as

$$\alpha \equiv \left| \frac{Z(I)}{Z(I_{\text{max}})} - 1 \right|$$

where $I_{\text{max}}$ is the maximum among all $I$. The quantity $\alpha$ may be termed a nonlinearity indicator because it vanishes for linear materials, while becoming non-vanishing and $I$-dependent for nonlinear materials.

At QNDE, 2010, C.C.H. Lo and the present author demonstrated successful applications of the nonlinear eddy current (NLEC) NDE to surface layer characterizations of ferromagnetic samples, where the nonlinearity indicator $\alpha$ was shown to correlate strongly with the sample conditions of interest. Specifically, the method was demonstrated to be effective, both on case-hardened steel samples, and on high-temperature-oxidized Ni-3Al alloy samples.

To understand the NLEC phenomenon quantitatively, the same QNDE talk presented a simplified NLEC theory based on a uniform incident field and layered half space. In general, Maxwell’s equations in a conducting medium read

$$\begin{align*}
\nabla \times \vec{E} + \dot{\vec{B}} &= 0 \\
\nabla \times \vec{H} &= \sigma \vec{E} - \vec{j}_c
\end{align*}$$

(2)
which do not close by themselves, until a constitutive B-H relation is given between the magnetic flux density \( \tilde{B} \) and the magnetic field \( \tilde{H} \). When the medium is a non-magnetic linear material, the linear relationship \( \tilde{B} = \mu \tilde{H} \) holds. For ferrous materials, there is an additional nonlinear term \( \tilde{B}_{NL} \), with which the constitutive relation reads

\[
\tilde{B} = \mu \tilde{H} + \tilde{B}_{NL}.
\]

For the case of uni-directional, uniform fields, and when the H field is weak, there exists the nonlinear hysteresis relation of Rayleigh’s, which reads [1,2]

\[
B_{NL}^\pm(t) = v_\alpha \left\{ H_m H(t) + \frac{1}{2} \left[ H_m^2 - H(t)^2 \right] \right\}
\]

where the field \( H(t) \) swings from \(-H_m\) to \(H_m\), and where the +[-] sign corresponds to the upper [lower] branch of the hysteresis loop. Rayleigh’s nonlinearity constant \( v_\alpha \) is a material parameter just as the initial permeability \( \mu_n \) is. The problem of a uniform incident field acting on a layered half space material was solved at the lowest nontrivial order of nonlinearity by the use of Born approximation. The model calculations reproduced the experimental results of the nonlinearity indicator \( \alpha \) qualitatively.

Successful NLEC model development for the case depth characterization problem motivated us to generalize the NLEC model to a broader class of EC NDE problems. In fact, a large majority of realistic EC inspections involves ferrite-coated probes which often exhibit nonlinear responses due to their ferrite nonlinearity. However, the Rayleigh formula (4) is inapplicable to general EC problems where the fields are neither uniform nor uni-directional. It is thus necessary to find a vector extension of the Rayleigh formula (4). This paper points out that the vector Preisach model with a constant weight function provides the needed vector extension. Moreover, the paper shows that it is possible to calculate the extended formula for a vector-valued harmonic motion as an input.

Following this Introduction, the next section presents a nonlinear B-H relation at the lowest order of nonlinearity in terms of the vector Preisach-Rayleigh formula, and its harmonic analysis. The subsequent section shows how to use the preceding result for solving the NLEC problem in Born approximation. The fourth section works out the NLEC problem in cylindrically symmetric geometry for concreteness, and presents numerical predictions of the nonlinear indicator \( \alpha \) due to a long rod with a case layer, before drawing several conclusions.

PREISACH-RAYLEIGH NONLINEAR HYSTERESIS FORMULA

This section presents derivation of a nonlinear hysteresis B-H relation à la Rayleigh’s, which is valid at the lowest nontrivial order of nonlinearity (Rayleigh regime) and applicable for vector-valued fields. We also conduct harmonic analysis of the resulting formula.

Extension of Rayleigh’s Formula to Two Dimensions

The first and critical step is to extend the Rayleigh nonlinear B-H formula (4) to a multi-dimensional formula, valid in the Rayleigh regime. For this purpose, we start with the scalar Preisach formula [3,4,5]

\[
B_{NL}(t) = \tilde{\gamma} H(t) = \int_{\alpha \geq \beta} w(\alpha, \beta) \tilde{\gamma}_{\alpha \beta} H(t) d\alpha d\beta
\]

where \( \gamma_{\alpha \beta} \) and \( w(\alpha, \beta) \) are “relay” operator and a weight function, respectively. This is a phenomenological model of hysteresis and, while being descriptive and flexible, it requires physics input to determine the weight function. It turned out that this determination is straightforward in the Rayleigh regime, i.e. the Rayleigh relation (4) follows Eq. (5) upon explicit integration when the weight function is set to be a constant, specifically \( w(\alpha, \beta) = v_\alpha / 2 \). We shall use this fact as a guiding principle, and derive the extension of uni-directional Rayleigh’s formula into a multi-dimensional formula, by starting from the vector version of the Preisach model, and by setting its weight function to be a constant.

In what follows, we work with the 2D version of the Preisach-Rayleigh formula, thus derived from the 2D vector Preisach model [5], which reads
\[
\vec{B}_{NL}(t) = \int_{-\pi/2}^{\pi/2} \vec{e}^{(\theta)} \hat{v} \int_{\alpha \geq 0 \beta} \hat{j}_{\alpha \beta} \left[ \vec{e}^{(\theta)} \cdot \vec{H}(t) \right] d\alpha d\beta d\theta \tag{6}
\]
where \( e^{(\theta)} = \hat{e}_x \cos \theta + \hat{e}_y \sin \theta \). The weight \( \vec{v} \) is taken to be a constant here, while it can be potentially \( \theta \)-dependent in case of material anisotropy.

**Harmonic Analysis of Vector Preisach Model**

The second important step is to conduct harmonic analysis of the nonlinear B-H relations. The uni-directional result was known earlier [e.g. Ref. 2], i.e. when the incident H field undergoes harmonic oscillation \( \vec{H}(t) = \vec{H} \cos(\omega t - \chi) \) with a frequency \( \omega \), the output B field \( B(t) \) is calculated through the Rayleigh formula (4) in terms of a Fourier series

\[
B_{NL}(t) = \text{Re} \sum_{n=1, \text{odd}}^\infty B_{NL}^{n0} e^{-in\omega t}, \quad B_{NL}^{n0} = n \vec{H}^2 e^{in\chi} \cdot c_n, \quad c_n = \begin{cases} 1 + \frac{1}{2n} & \text{if } n = 1 \\ -i \frac{1}{\pi n(p^2-1)} & \text{if } n = 3, 5, \ldots \end{cases} \tag{7}
\]
The corresponding harmonic analysis in 2-D can be carried out in a closed form as well. Consider the input harmonic motion of the form

\[
\begin{align*}
H_x(t) &= \vec{H}_x \cos(\omega t - \chi_x) \\
H_y(t) &= \vec{H}_y \cos(\omega t - \chi_y)
\end{align*} \tag{8}
\]
where the tip of the H vector traces an elliptical locus. When this motion is inserted into the 2D Preisach-Rayleigh formula (6) with a constant weight \( \vec{v} \), the output B field can be calculated into the form

\[
\begin{bmatrix} B_{NL,x} \\ B_{NL,y} \end{bmatrix}(t) = \text{Re} \sum_{n=1, \text{odd}}^\infty \begin{bmatrix} B_{NL,x}^{n0} \\ B_{NL,y}^{n0} \end{bmatrix} e^{-in\omega t}. \tag{9}
\]
When \( \chi_x = \chi_y \), the motion reduces to a harmonic oscillation, while the Preisach-Rayleigh formula yields \( B_{NL}^{n0} = (8/3 \vec{H}^2 e^{in\chi} \cdot c_n \), namely \( \vec{v} = (3/8) \vec{H}_x \). In general, the \( B_{NL} \) fields are expressed in terms of a finite sum of the Gauss hypergeometric function \( _2F_1(\alpha, \beta; \gamma; z) \). Explicitly,

\[
B_{NL,x}^{n0} = B_{NL,y}^{n0} \cos \theta_0 + e^{2i\chi} B_{NL,y}^{n0} \sin \theta_0 = 2\sqrt{\pi} \vec{v} \vec{H}^2 e^{in\chi} \cdot c_n \left( 1 + e^{2i\chi} - 1 \right) \sin^2 \theta_0 \frac{n}{2} \tag{10}
\]

where \( \vec{H}_x = \vec{H} \cos \theta_0, \vec{H}_y = \vec{H} \sin \theta_0, \chi = \chi_y - \chi_x \), and where \( n_0 = n \pm 1 \), \( \rho = \sqrt{1 - \sin^2 2\theta_0 \sin^2 \chi} \).

**SOLUTION PROCEDURE VIA BORN APPROXIMATION**

Given the method to calculate the B-H relation, we can proceed to solve Maxwell’s equations (2) for each frequency \( \omega \) in the lowest order of nonlinearity by the use of Born approximation. Specifically, we split the total \( B \) field into the linear and nonlinear terms according to Eq. (3), and cast Eq. (2) into
\begin{equation}
\begin{aligned}
\tilde{\nabla} \times \tilde{E}^{\omega} - i \omega \mu_n \tilde{H}^{\omega} &= i \omega B_{NL}^{\omega}, \\
\tilde{\nabla} \times \tilde{H}^{\omega} - \sigma E^{\omega} &= j_e^{\omega}.
\end{aligned}
\end{equation}

(11)

The solution process in Born approximation allows us to treat the $-i\omega B_{NL}$ term as a magnetic current source $j_e^{\omega}$. Indeed, the solution process proceeds as follows:

1. Turn the nonlinear term off, and calculate the 0\textsuperscript{th} order fields for a given set of coil currents as a usual linear problem.
2. Insert the resulting 0-th order H field into the Preisach-Rayleigh formula, and calculate the 1\textsuperscript{st} order $B_{NL}$.
3. Calculate the 1\textsuperscript{st} order fields as the second linear problem, with $-i\omega B_{NL}$ used as the magnetic current source.
4. Finally calculate the total field as a sum of the 0\textsuperscript{th} and 1\textsuperscript{st} order fields, and/or the 0\textsuperscript{th} and 1\textsuperscript{st} order impedances, $Z^{(0)}$ and $Z^{(1)}$ respectively, as needed. For example, the nonlinearity indicator $\alpha$ can be calculated as

\begin{equation}
\alpha = \left| \frac{Z^{(1)}(I_{\text{drive}})}{Z^{(0)}} \right|
\end{equation}

within our approximation.

**NONLINEAR EC MODELING FOR LAYERED CYLINDRICAL ROD**

In this section, we will exercise the solution procedure described in the preceding section for a prototypical EC problem, i.e. finite encircling coils wound around a cylindrical rod of an infinite length. Specifically considered is a layered cylindrical rod consisting of a layer formed around a core, where the core is made of a nonlinear material “a” (e.g. steel), while the layer is made of a linear material “b” (e.g. hardened steel that is approximately linear) [FIGURE 1].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Illustration of the model configuration, with circular coil(s) encircling a layered cylindrical rod, where the core region I is made of a nonlinear material “a”, while the layer region II is made of a linear material “b”.}
\end{figure}

**Fields in the Layered Cylinder**

We first obtain the 0\textsuperscript{th} order fields after setting $v_k=0$. The problem reduces to the well-known linear problem [e.g. Ref. 6], and the field solutions can be written in terms of Fourier components $p_z$ in the axial direction $z$. Explicitly, the fields are expressed in terms of the modified Bessel functions as
\[ H_{r}^{\alpha,p}(r,z) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipz} \left[ H_{r}^{\alpha-p}(r) \right] \]  \\
\[ H_{z}^{\alpha,p}(r,z) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipz} \left[ H_{z}^{\alpha-p}(r) \right] \]  

(13a)

\[ \left[ \begin{array}{c} \left[ H_{r}^{\alpha,p}(r) \right]^{(0)} \\ \left[ H_{z}^{\alpha,p}(r) \right]^{(0)} \end{array} \right] = \begin{bmatrix} -\frac{i}{\rho_{0}} I_1(\gamma_{d}r) \\ -Y_{a}I_{0}(\gamma_{d}r) \end{bmatrix} / I_{1}(\xi) \begin{bmatrix} D_{1} - Y_{a}K_{0}(\xi_{0}I)U_{1}^{-1} \xi_{0}K_{1}(\xi_{0}) \end{bmatrix}^{-1} K^{c}(p_{z}, \{a_{i}, h_{j}\}) \]  

(13b)

\[ \left[ \begin{array}{c} U_{1} \\ D_{1} \end{array} \right] = P \left[ \begin{array}{c} I_{1}(\xi_{b1}) \\ -Y_{b}I_{0}(\xi_{b1}) \end{array} \right] + Q \left[ \begin{array}{c} K_{1}(\xi_{b1}) \\ Y_{b}K_{0}(\xi_{b1}) \end{array} \right] \]  

(13c)

\[ K^{c}(p_{z}, \{a_{i}, h_{j}\}) = \sum_{i} \left[ p_{z} \{a_{i}, K_{1}(p_{z}, \{a_{i}\}) \} \right] \sum_{j} e^{-ip, h_{j}} \]  

(13d)

where we introduced several notations:

\[ \gamma_{d} \equiv \left( p_{z}^{2} - k_{a}^{2} \right)^{1/2}, \quad Y_{a} \equiv \gamma_{a} / (-\mu_{a}^{*}), \quad Z_{a} \equiv Y_{a}^{-1}, \quad \xi_{a2} = \gamma_{a} \mu_{a}^{*}, \quad I_{m/n}(z) \equiv I_{m}(z) / I_{n}(z), \quad \text{etc.} \]  

(14)

The expression for the 0-th order impedance \( Z^{(0)} \) can be obtained similarly.

We next apply the 2D harmonic analysis of the preceding section, and compute the first order nonlinear B field \( B_{N1}^{(1)ns}(r,z) \) from the 0-th order field \( H^{(0)ns}(r,z) \) through the 2D Preisch-Rayleigh formula (10). The results are transformed back to the Fourier space for the subsequent use.

We then obtain the first order solutions by solving the second linear problem, using \( j_{m}^{(1)ns} \) as the source this time. The first order surface field and the impedance can be obtained accordingly. In the same notation, they read

\[ E_{\phi}^{(1)ns,p}(r_{1}) = (-i \omega \mu_{0}) \left[ I_{b}(\xi_{a2}) \left[ I_{0}(\xi_{a2}) \right] \right]^{-1} \sum_{n} \left[ p_{z} \{a_{i}^{R}, h_{j}^{R} \} \right] \left[ \frac{1}{i \omega \mu_{0}} \right] E_{\phi}^{(1)ns,p}(r_{1}) \]  

(15a)

\[ \left[ \begin{array}{c} \hat{U} \\ \hat{D} \end{array} \right] = P \left[ \begin{array}{c} I_{1}(\xi_{b2}) \\ -Y_{b}I_{0}(\xi_{b2}) \end{array} \right] + Q \left[ \begin{array}{c} K_{1}(\xi_{b2}) \\ Y_{b}K_{0}(\xi_{b2}) \end{array} \right] \equiv \xi_{b2} \left[ \begin{array}{c} K_{0}(\xi_{b2}) \\ Y_{b}Z_{a}K_{0}(\xi_{b2}) \end{array} \right] \]  

(15b)

\[ I_{a}^{(1)ns,p}(r_{2},0) = \sum_{n} \left[ p_{z} \{a_{i}^{R}, h_{j}^{R} \} j_{m,r}^{(0)}(r_{1}) + \gamma_{a} \gamma_{a}^{*} \right] \left[ \frac{1}{i \omega \mu_{0}} \right] E_{\phi}^{(1)ns,p}(r_{1}) \]  

(15c)

\[ K^{c}(p_{z}, \{a_{i}, h_{j}^{R} \}) = \sum_{i} \left[ p_{z} \{a_{i}, K_{1}(p_{z}, \{a_{i}\}) \} \right] \sum_{j} e^{-ip, h_{j}^{R}} \]  

(15d)

**Nonlinearity Indicator: Computed**

From the mathematical expressions in the preceding subsection, we can compute impedances in the 0-th and first orders, and hence the nonlinearity indicator according to Eq. (12). It is a straightforward numerical task, once we stabilize the computation by isolating exponential behaviors from the modified Bessel functions [7]. We have chosen a case hardened layer characterization problem as an example, and computed the nonlinearity indicator as the function of relative drive current intensity. To be specific, the computation has been carried out for the maximum
current intensity $I_{\text{max}}$ of 0.1 A, and for the case depths ranging between 0 mm and 4 mm, with the parameters tabulated in **TABLE 1** below. **FIGURE 2** presents a typical result, computed at 32 Hz.

**TABLE 1**: Parameters and their values used in the example computation.

<table>
<thead>
<tr>
<th>Parameter Type</th>
<th>Parameter Variable</th>
<th>Parameter Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive Coil Radius</td>
<td>$a^D$</td>
<td>16-18 [mm]</td>
</tr>
<tr>
<td>Drive Coil Length</td>
<td>$l^D$</td>
<td>2 [mm]</td>
</tr>
<tr>
<td>Drive Coil, # Turns</td>
<td>$n^D$</td>
<td>110</td>
</tr>
<tr>
<td>Receive Coil Radius</td>
<td>$a^R$</td>
<td>15.5-15.8 [mm]</td>
</tr>
<tr>
<td>Receive Coil Length</td>
<td>$l^R$</td>
<td>1 [mm]</td>
</tr>
<tr>
<td>Receive Coil, # Turns</td>
<td>$n^R$</td>
<td>20</td>
</tr>
<tr>
<td>Cylinder diameter</td>
<td>$2r_1$</td>
<td>30 [mm]</td>
</tr>
<tr>
<td>Initial permeability, case</td>
<td>$\mu_{\text{in,case}}$</td>
<td>61.6</td>
</tr>
<tr>
<td>Conductivity, case</td>
<td>$\sigma_{\text{case}}$</td>
<td>7.05 [%IACS]</td>
</tr>
<tr>
<td>Initial permeability, core</td>
<td>$\mu_{\text{in,core}}$</td>
<td>110.5</td>
</tr>
<tr>
<td>Conductivity, core</td>
<td>$\sigma_{\text{core}}$</td>
<td>8.86 [%IACS]</td>
</tr>
<tr>
<td>Nonlinearity</td>
<td>$\gamma_{\text{core}}$</td>
<td>5.93 [1/Oe]</td>
</tr>
</tbody>
</table>

**FIGURE 2**: Computed nonlinearity indicator $\alpha$ plotted against the drive current intensity ratio $I/I_{\text{max}}$ in the dB unit, for the case depths of (a) 0 mm (core only), (b) 1 mm, (c) 2 mm, (d) 3 mm, and (e) 4 mm, driven at 32 Hz.

**CONCLUSIONS**

In conclusion, it is shown that the vector Preisach model provides a basis for formulating nonlinear eddy current (NLEC) modeling. Specifically, we have found an extension of Rayleigh’s B-H relation to a vector-valued nonlinear hysteresis formula at the lowest nontrivial order of nonlinearity, by way of choosing a constant weight function in the vector Preisach model. Second, we have shown how to calculate the resulting vector B-H relation explicitly for the input H field undergoing vector harmonic motion, and obtained analytical expressions of the output nonlinear B field $\mathbf{B}_{\text{NL}}$, expressed in terms of a finite sum of hypergeometric functions. Third, we have presented a solution procedure of the NLEC problem in Born approximation, by solving linear problems twice at the 0th and 1st orders of nonlinearity, while using the B-H relation to link the 0th and 1st order solutions. Fourth, we have explicitly exercised the solution procedure via Born approximation for a layered cylinder problem, predicting the nonlinearity indicator arising from a long case-hardened rod.

It should be remarked that detailed mathematical descriptions are needed for the complete exposition of this NLEC modeling method, the task being deferred to a publication elsewhere.
ACKNOWLEDGMENTS

The author wishes to dedicate this paper to the memory of the late Professor Donald O. Thompson, who has been the leader, teacher, and source of encouragement for him. This work was supported by the NSF Industry/University Cooperative Research Program of the Center for Nondestructive Evaluation at Iowa State University.

REFERENCES